

PHYSICS 225 A HOMEWORK 3

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(1) Definition of group given in class was

$$\forall a, b \in G, \Rightarrow ab \in G$$

$$(ab)c = a(bc)$$

$$\exists e \in G, \forall a \in G, ae = ea = a$$

$$\forall a \in G, \exists a^{-1} \in G, aa^{-1} = a^{-1}a = e$$

(a) Assume that we have two identities e_1 and e_2

$$\begin{aligned} \text{Then } e_1 e_2 &= e_1 \\ e_1 e_2 &= e_2 \end{aligned} \Rightarrow e_1 = e_2$$

(b) Assume that a has two inverses a^{-1} and $\tilde{a}^{-1}$

$$a^{-1} = \tilde{a}^{-1}e = \tilde{a}^{-1}(a\tilde{a}^{-1}) = (\tilde{a}^{-1}a)\tilde{a}^{-1} = \tilde{a}^{-1}$$

$$(c) (\tilde{a}^{-1})^{-1} a^{-1} = e$$

Multiply from the right by e

$$(\tilde{a}^{-1})^{-1} (\tilde{a}^{-1}a) = (ea) \Rightarrow (\tilde{a}^{-1})^{-1} = a$$

$$(d) (b^{-1}\tilde{a}^{-1})(ab) = b^{-1}(\tilde{a}^{-1}a)b = b^{-1}b = e$$

② Let $\vec{r}$ be eigenvector of A in L with eigenvalue λ

$$A \vec{r} = \lambda \vec{r}$$

Consider $\vec{r}'_a = G(a) \vec{r}$

Then ~~Also~~ $A \vec{r}'_a = A G(a) \vec{r} = G(a) A \vec{r} = G(a) [\lambda \vec{r}]$

$$A \vec{r}'_a = \lambda G(a) \vec{r} = \lambda \vec{r}'_a$$

So $\vec{r}'_a$ is eigenvector of A with the same eigenvalue λ

The set of vectors $\vec{r}'_a$ generated as $G(a)$ runs through all elements of G form an invariant subspace because

$$G(b) \vec{r}'_a = G(b) G(a) \vec{r} = G(ab) \vec{r} = G(c) \vec{r} = \vec{r}'_c$$

$\uparrow$
 $c = a \cdot b$

But L is irreducible $\Rightarrow$ set of vectors $\vec{r}'_a$ span entire vector space $L \Rightarrow$ Any vector in L is superposition of $\vec{r}'_a$

Since $A \vec{r}'_a = \lambda \vec{r}'_a$ ~~must be~~

Then $\forall \vec{x} \in V \quad A \vec{x} = \lambda \vec{x} \Rightarrow (A = \lambda I)$

Abelian group $G(a)G(b) = G(b)G(a)$

From Schur's 1st lemma it follows that $G(a) = aI$
The only way for the representation to be irreducible is for its dimension to be one

$$③ \quad J^2 |j m\rangle = j(j+1) |j m\rangle$$

$$J_3 |j m\rangle = m |j m\rangle$$

$$J_{\pm} = J_1 \pm i J_2$$

$$\text{Also } \pm J_{\pm} = [J_3, J_{\pm}] = J_3 J_{\pm} - J_{\pm} J_3 \Rightarrow \underline{J_3 J_{\pm} = J_{\pm} J_3 \pm J_{\pm}}$$

$$\begin{aligned} \text{Consider } J_3 (J_{\pm} |j m\rangle) &= J_{\pm} J_3 |j m\rangle \pm J_{\pm} |j m\rangle \\ &= (m \pm 1) J_{\pm} |j m\rangle \end{aligned}$$

$$\text{Therefore } J_+ |j m\rangle = C_+^m |j, m+1\rangle$$

$$J_- |j m\rangle = C_-^m |j, m-1\rangle$$

where C_+^m and C_-^m are some constants -

The representation is irreducible

There must be $m = \max$ such that $J_+ |j, \max\rangle = 0$

In principle there could be more than one eigenvector of J_3 with eigenvalue m - So I will introduce another label " i "

$$|j m\rangle \rightarrow |j m i\rangle$$

Now consider $|j, \max, i\rangle$

If I act on this state with J_- , I will get some number times an eigenvector of J_3 with eigenvalue ~~(max)~~ $(\max-1)$ - Call this $|j, \max-1, i\rangle$

Now apply J_- repeatedly - Eventually I will get zero, otherwise the dimensionality is infinite.

$$J_- |j, \min i\rangle = 0$$

Now apply J_+ to $|j, \min i\rangle$:

$$\text{Remember } |j, \min i\rangle = \underset{\substack{\uparrow \\ \text{Some} \\ \text{number}}}{C} J_- |j, \min+1 i\rangle$$

$$J_+ |j, \min i\rangle = C J_+ J_- |j, \min+1 i\rangle$$

$$\text{But } J_+ J_- = J^2 - J_z^2 + J_z, \text{ so}$$

$$J_+ J_- |j, \min+1 i\rangle = C [l(l+1) - \min^2 + \min] |j, \min+1 i\rangle$$

So, not only I get back an eigenvector of J_z with $m = \min+1$, but I get the same eigenvector which when operated on with J_- gives me $|j, \min i\rangle$ (i.e. the extra label " i " is the same)

I can do this over and over and show that

$$|j, \max i\rangle, |j, \max-1 i\rangle, \dots, |j, (\min+1) i\rangle, |j, \min i\rangle$$

is a closed subspace with respect to the application of the operators J_z, J_+, J_- .

But since $J_{\pm} = J_1 \pm i J_2$ this is also closed with respect to J_3, J_1, J_2 .

Now assume there is another eigenvector

$$J_3 |j m i'\rangle = m |j m i'\rangle \quad \underline{i' \neq i}$$

If this exists, representation is reducible

IN AN IRREDUCIBLE REPRESENTATION THE EIGENVECTORS $|j m\rangle$ MUST BE UNIQUE

N

$$J_{\pm} |j m\rangle = C_{\pm}^m |j m \pm 1\rangle$$

We now want to find C_{+}^m

$$\text{Consider } |\psi\rangle = J_{+} |j m\rangle = C_{+}^m |j m+1\rangle$$

$$\text{Then } \langle\psi|\psi\rangle = |C_{+}^m|^2$$

But also

$$\langle\psi|\psi\rangle = \langle\psi|J_{+}^{\dagger}J_{+}|\psi\rangle = \langle\psi|J_{-}J_{+}|\psi\rangle$$

$$\langle\psi|\psi\rangle = \langle\psi|J^2 - J_3^2 - J_3|\psi\rangle$$

$$\langle\psi|\psi\rangle = j(j+1) - m^2 - m$$

$$\langle\psi|\psi\rangle = j(j+1) - m(m+1) = |C_{+}^m|^2$$

$$\Rightarrow C_{+}^m = e^{i\phi} \sqrt{j(j+1) - m(m+1)}$$

Take $\phi=0$ by redefining the basis vector ~~$e^{i\phi}$~~

Similarly, starting from $|\psi\rangle = J_{-} |j m\rangle$ can show

$$C_{-}^m = \sqrt{j(j+1) - m(m-1)}$$

Now let's find max and min

$$J_+ |j \max\rangle = 0 \Rightarrow j(j+1) - \max(\max+1) = 0$$

This has two solutions max = j & max = -(j+1)

$$J_- |j \min\rangle = 0 \Rightarrow j(j+1) - \min(\min-1) = 0$$

This has two solutions min = -j & min = j+1

Since $\max > \min$ we have two cases

CASE 1 $j > 0$, then $\max = j$ $\min = -j$

CASE 2 $j < 0$, then $\max = -(j+1)$ $\min = j+1$
Relabel this by $l = -(j+1)$, i.e. $j = -(l+1)$

Then $l > 0$, $\max = l$ $\min = -l$

Also eigenvalue of J^2 is unchanged

$$j(j+1) = -(l+1)l = l(l+1)$$

So CASE 2 = CASE 1

Then, look at **CASE 1**

$$j > 0 \quad \max = j \quad \min = -j \Rightarrow m = j, j-1, j-2, \dots, -j$$

j must be integer or half integer

$$④ \quad G(\vec{a}) = \exp(i\vec{a} \cdot \vec{X})$$

Infinitesimal

$$G(\vec{a}) \sim 1 + i a_k X_k - \frac{1}{2} (\sum a_k X_k)^2$$

$$G(\vec{b}) \sim 1 + i b_e X_e - \frac{1}{2} (b_e X_e)^2$$

$$G^{-1}(\vec{a}) \approx 1 - i a_m X_m - \frac{1}{2} (a_m X_m)^2$$

This is true because

$$G(\vec{a}) G^{-1}(\vec{a}) = \left[1 + i a_k X_k - \frac{1}{2} (a_k X_k)^2 \right] \cdot$$

$$\left[1 - i a_m X_m - \frac{1}{2} (a_m X_m)^2 \right]$$

$$= \left[1 - i (a_m X_m) - \frac{1}{2} (a_m X_m)^2 \right.$$

$$+ i (a_k X_k) + (a_k X_k)(a_m X_m)$$

$$\left. - \frac{1}{2} (a_k X_k)^2 + \text{higher order} \right] = 1$$

$$\text{So } G(\vec{a}) G(\vec{b}) G^{-1}(\vec{a}) G^{-1}(\vec{b}) \approx$$

$$\left[1 + i a_k X_k - \frac{1}{2} (a_k X_k)^2 \right] \times$$

$$\left[1 + i b_e X_e - \frac{1}{2} (b_e X_e)^2 \right] \times$$

$$\left[1 - i a_m X_m - \frac{1}{2} (a_m X_m)^2 \right] \times$$

$$\left[1 - i b_n X_n - \frac{1}{2} (b_n X_n)^2 \right]$$

I will get something like $1 + O(X) + O(X_i X_j)$

The first order term is clearly zero -

The second order term is

$$2 \times \left[-\frac{1}{2} (a_k X_k)^2 \right] + 2 \left[-\frac{1}{2} (b_n X_n)^2 \right]$$

$$i (a_k X_k) \left[i (b_e X_e) - i (a_m X_m) - i (b_n X_n) \right] +$$

$$i (b_e X_e) \left[-i (a_m X_m) - i (b_n X_n) \right] +$$

$$- i (a_m X_m) \left[-i b_n X_n \right]$$

$$= - (a_k X_k) (b_e X_e) + (a_k X_k) (b_n X_n) + (b_e X_e) (a_m X_m) - (a_m X_m) (b_n X_n)$$

$$= a_m b_e [X_e X_m]$$

$$\text{So } G(\vec{a}) G(\vec{b}) G^{-1}(\vec{a}) G^{-1}(\vec{b}) \approx 1 + \sum_{me} a_m b_e [X_e X_m]$$

But, by the closure of the group

$$G(c) = G(\vec{a}) G(\vec{b}) G^{-1}(\vec{a}) G^{-1}(\vec{b}) \approx 1 + \sum c_k X_k$$

$$\Rightarrow \boxed{\sum_k c_k X_k = \sum_{me} a_m b_e [X_e X_m]} \quad \text{eqn 1}$$

$$\text{But also } \vec{c} = \vec{f}(\vec{a}, \vec{b})$$

Expand in Taylor series

$$c_i = f_i(0, 0) + \frac{\partial f_i}{\partial a_j} a_j + \frac{\partial f_i}{\partial b_j} b_j + \frac{1}{2} \sum_{jk} \frac{\partial^2 f_i}{\partial a_j \partial a_k} a_j a_k$$

$$\quad \quad \quad \uparrow$$

$$= 1 + \frac{1}{2} \sum_{jk} \frac{\partial^2 f_i}{\partial b_j \partial b_k} b_j b_k + \frac{1}{2} \sum_{jk} \frac{\partial^2 f_i}{\partial a_j \partial b_k} a_j b_k$$

But $\vec{f}(\vec{a}, \vec{b}) = f(\vec{a}, \vec{b}) = f(\vec{b}, \vec{a}) = \vec{0}$

$$c_i = \frac{1}{2} \sum_{jk} \frac{\partial^2 f_i}{\partial a_j \partial b_k} a_j b_k = \sum_{jk} \alpha_{jk}^i a_j b_k$$

↑ this defines α_{jk}^i

So, comparing to equation (1)

$$\sum_{m \neq k} \alpha_{me}^k a_m b_e X_k = \sum_{m \neq e} a_m b_e [X_e X_m]$$

Because the Taylor expansion is unique

$$[X_e X_m] = \sum_k \alpha_{me}^k X_k$$

⑤ $[X_i X_j] = \sum_k c_{ij}^k X_k \quad g_{ij} = \sum_{ts} c_{it}^s c_{js}^t$

$$C = \sum_{pq} (g^{-1})_{pq} X_p X_q$$

$$[C X_a] = \sum_{pq} (g^{-1})_{pq} [X_p X_q X_a]$$

we want to show that this is zero, $\forall a$

$$[X_p X_q X_a] = X_p [X_q X_a] + [X_p X_a] X_q$$

$$\begin{aligned} \text{Then } [C X_a] &= \sum_{pq} (g^{-1})_{pq} \{ X_p [X_q X_a] + [X_p X_a] X_q \} \\ &= \sum_{pq} (g^{-1})_{pq} \left\{ \sum_r c_{qa}^r X_p X_r + \sum_r c_{pa}^r X_r X_q \right\} \end{aligned}$$

Since I am summing over p, q, r let me exchange labels on the second sum

$$\begin{aligned} r &\rightarrow p \\ q &\rightarrow r \\ p &\rightarrow q \end{aligned}$$

$$[C X_a] = \sum_{pqr} (g^{-1})_{pq} C_{qa}^r X_p X_r + \sum_{pqr} (g^{-1})_{qr} C_{qa}^p X_p X_r$$

Define $A_{pr}^a = \sum_q (g^{-1})_{pq} C_{qa}^r$ $B_{pr}^a = \sum_q (g^{-1})_{qr} C_{qa}^p$

$$[C X_a] = \sum_{pr} A_{pr}^a X_p X_r + \sum_{pr} B_{pr}^a X_p X_r$$

But $B_{pr}^a = \sum_q (g^{-1})_{qr} C_{qa}^p = \sum_q (g^{-1})_{rq} C_{qa}^p = A_{rp}^a$
 $\uparrow$
 g^{-1} is symmetric

$$\text{So } [C X_a] = \sum_{pr} (A_{pr}^a + A_{rp}^a) X_p X_r$$

This is zero if $A_{pr}^a = -A_{rp}^a$

I will show that $D = gAg$ is antisymmetric

Note If A is antisymmetric ~~and~~ and g is symmetric (it is!) then D is antisymmetric

Proof $D_{ij} = \sum_{ke} g_{ik} A_{ke} g_{ej} = \sum_{ke} g_{ki} (-A_{ek}) g_{je} =$

$$D_{ij} = - \sum_{ke} g_{je} A_{ek} g_{ki} = -D_{ij} \quad \checkmark$$

Conversely if D is antisymmetric, A must be ~~symmetric~~ antisymmetric

$$D = g A g$$

$$D g^{-1} = A$$

$$g^{-1} D g^{-1} = A$$

Since g^{-1} is also symmetric, I can use the same argument above to show that A is antisymmetric

END OF NOTE

OK Let's get back to $D^a = g A^a g$

$$D_{ij}^a = \sum_{pr} g_{ip} A_{pr}^a g_{rj} = \sum_{pqr} \underbrace{(g_{ip})(g^{-1})_{pq}}_{\delta_{iq} \text{ when summed over } p} c_{qa}^r g_{rj}$$

$\uparrow \delta_{iq}$ when summed over p

$$D_{ij}^a = \sum_{qr} \delta_{iq} c_{qa}^r g_{rj} = \sum_r c_{ia}^r g_{rj}$$

But $g_{rj} = \sum_{st} c_{rs}^t c_{jt}^s$

$$D_{ij}^a = \sum_{rst} c_{ia}^r c_{rs}^t c_{jt}^s \quad \text{EQUATION 1}$$

Consider the Jacobi identity

$$[X_a [X_b X_c]] + [X_b [X_c X_a]] + [X_c [X_a X_b]] = 0$$

$$[X_a \sum_e c_{bc}^e X_e] + [X_b \sum_e c_{ca}^e X_e] + [X_c \sum_e c_{ab}^e X_e]$$

$$\sum_m (C_{bc}^e C_{ae}^m + C_{ca}^e C_{be}^m + C_{ab}^e C_{ce}^m) X_m = 0$$

But the generators are linearly independent, therefore

$$\boxed{\sum_e C_{bc}^e C_{ae}^m + C_{ca}^e C_{be}^m + C_{ab}^e C_{ce}^m = 0} \quad \text{Eqtn 2}$$

Let's go back to eqtn 1

$$D_{ij}^a = \sum_{rst} C_{ia}^r C_{rs}^t C_{jt}^s = - \sum_{rst} C_{ia}^r C_{sr}^t C_{jt}^s$$

(because c's are antisymmetric under exchange of lower indices)

Now let's do the sum over r using eqtn 2

$$\begin{array}{lll} \text{Indices} & l \leftrightarrow r & c \leftrightarrow a \quad a \leftrightarrow s \\ & b \leftrightarrow i & m \leftrightarrow t \end{array}$$

$$D_{ij}^a = \sum_{rst} (+ C_{as}^r C_{ir}^t C_{jt}^s - C_{is}^r C_{ar}^t C_{jt}^s)$$

and

$$D_{ji}^e = \sum_{rst} (+ C_{as}^r C_{jr}^t C_{jt}^s + C_{sj}^r C_{ra}^t C_{ti}^s)$$

I am summing over all r, s, t

I am free to change labels as I please

For the 1st term $t \rightarrow s \quad s \rightarrow r \quad r \rightarrow t$

For the 2nd term $t \rightarrow r \quad s \rightarrow t \quad r \rightarrow s$

$$D_{ji}^a = \sum_{rst} C_{ar}^t C_{jt}^s C_{is}^r + C_{tr}^s C_{sa}^r C_{ri}^t$$

$$D_{ij}^a = \sum_{rst} C_{ar}^t C_{jt}^s C_{is}^r - C_{jt}^s C_{as}^r C_{ir}^t = -D_{ji}^a$$

$\Rightarrow D^a$ is antisymmetric $\Rightarrow A^a$ is antisymmetric $\Rightarrow [X_i, A^a] = 0$

⑥ $\pi^- d \rightarrow nn$ (strong interaction, conserves parity)

Initial state, angular momentum $J=1$

Final state $J=L+S$, must add up to $J=1$

Look at the sym. properties of $|nn\rangle$ under particle interchange

	spin	spatial	total	allowed
$J=1 \quad L=2 \quad S=1$	+	+	+	NO
$J=1 \quad L=1 \quad S=1$	+	-	-	YES
$J=1 \quad L=1 \quad S=0$	-	-	+	NO
$J=1 \quad L=0 \quad S=1$	+	+	+	NO

$\Rightarrow L=1 \Rightarrow$ Parity of final state $= (-1)^L \cdot (\text{Parity of } n)^2 = (-1)^1 = -1$

Parity of initial state $= (P_{\pi^-}) \cdot (-1)^L \cdot (P_d)$

$= P_{\pi^-} \cdot 1 \cdot 1 = P_{\pi^-}$

$\Rightarrow P_{\pi^-} = -1$

7 (a) π^0 in $|I I_3\rangle = |1 0\rangle$

$$|\pi^0 \pi^0\rangle = |1 0\rangle |1 0\rangle = \sqrt{\frac{2}{3}} |2 0\rangle - \sqrt{\frac{1}{3}} |0 0\rangle$$

using tables of Clebsch-Gordan coeff.

$$\Rightarrow \boxed{I = 0, 2}$$

(b) Wavefunction must be sym.

$I = 0, 2$ states are symmetric

$\Rightarrow L$ must be even

(c) ρ^0 has $J=1 \Rightarrow \rho^0 \rightarrow \pi^0 \pi^0$ forbidden

since $J = L(\pi^0 \pi^0)$ because π^0 is spinless -

8 (a) $\pi^+ \pi^- \pi^0 : |1 1\rangle |1 -1\rangle |1 0\rangle$

$$|1 1\rangle |1 -1\rangle = \frac{1}{\sqrt{6}} |2 0\rangle + \frac{1}{\sqrt{2}} |1 0\rangle + \frac{1}{\sqrt{3}} |0 0\rangle$$

$$\text{Then } |2 0\rangle |1 0\rangle = \sqrt{\frac{3}{5}} |3 0\rangle - \sqrt{\frac{2}{5}} |1 0\rangle$$

$$|1 0\rangle |1 0\rangle = \sqrt{\frac{2}{3}} |2 0\rangle - \sqrt{\frac{1}{3}} |0 0\rangle$$

$$|0 0\rangle |1 0\rangle = |1 0\rangle$$

$$\Rightarrow \boxed{I = 0, 1, 2, 3}$$

$$(b) \pi^0 \pi^0 \pi^0 \quad |10\rangle |10\rangle |10\rangle$$

$$|10\rangle |10\rangle = \sqrt{\frac{2}{3}} |20\rangle - \sqrt{\frac{1}{3}} |00\rangle$$

$$\text{Then} \quad |20\rangle |10\rangle = \sqrt{\frac{3}{5}} |30\rangle - \sqrt{\frac{2}{5}} |10\rangle$$

$$|00\rangle |10\rangle = |10\rangle$$

$$\Rightarrow \underline{1, 3} \Rightarrow \boxed{I=1, 3}$$

$$(9) \quad J_+ = J_1 + iJ_2$$

$$J_- = J_1 - iJ_2 \Rightarrow J_2 = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{for } j = \frac{1}{2}$$

$$\text{Then} \quad e^{-i\theta J_2} = 1 + (-i\theta J_2) + \frac{(-i\theta J_2)^2}{2!} + \frac{(-i\theta J_2)^3}{3!} + \dots$$

$$iJ_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(iJ_2)^2 = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{1}{4} I = -\frac{1}{2^2} I$$

$$(iJ_2)^3 = -\frac{1}{2^3} (iJ_2)$$

$$\text{and so on} \Rightarrow \text{for } N\text{-even} \quad (iJ_2)^N = \frac{1}{2^N} (-1)^{N/2} I$$

$$\text{for } N\text{-odd} \quad (iJ_2)^N = \frac{1}{2^N} (-1)^{N/2} (iJ_2)$$

Now back to $e^{-i\theta J_2}$

0

$$e^{-i\theta J_z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\theta/2 \\ \theta/2 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} -(\theta/2)^2 & 0 \\ 0 & -(\theta/2)^2 \end{pmatrix} + \dots$$

$$e^{-i\theta J_z} = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} \quad \checkmark$$

Now consider $j=1$ case

$$\begin{aligned} J_z |1 1\rangle &= \frac{1}{2i} (J_+ |1 1\rangle - J_- |1 1\rangle) \\ &= \frac{1}{2i} (-\sqrt{2}) |1 0\rangle = -\frac{\sqrt{2}}{2i} |1 0\rangle \end{aligned}$$

$$\begin{aligned} J_z |1 0\rangle &= \frac{1}{2i} (J_+ |1 0\rangle - J_- |1 0\rangle) \\ &= \frac{\sqrt{2}}{2i} |1 1\rangle - \frac{\sqrt{2}}{2i} |1 -1\rangle \end{aligned}$$

$$J_z |1 -1\rangle = \frac{1}{2i} \sqrt{2} |1 0\rangle$$

$$\Rightarrow J_z = \frac{\sqrt{2}}{2i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$i J_z = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{pmatrix}$$

$$(iJ_2)^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$(iJ_2)^3 = \frac{1}{2\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -iJ_2$$

Back to $e^{-i\theta J_2} = 1 + (-i\theta J_2) + \frac{(-i\theta J_2)^2}{2!} + \dots$

odd powers $iJ_2 \left(-\theta + \frac{\theta^3}{3!} - \dots \right) = iJ_2 \sin \theta$

even powers $I + (iJ_2)^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots \right) = I + (iJ_2)^2 (1 - \cos \theta)$

So, finally

$$e^{-i\theta J_2} = I + \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} (1 - \cos \theta) + \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \sin \theta$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2} - \frac{1}{2} \cos \theta \\ \frac{1}{\sqrt{2}} \sin \theta & \cos \theta & -\frac{1}{\sqrt{2}} \sin \theta \\ \frac{1}{2} - \frac{1}{2} \cos \theta & \frac{1}{\sqrt{2}} \sin \theta & \frac{1}{2} + \frac{1}{2} \cos \theta \end{pmatrix}$$

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(a) Consider $|I, I_3=0\rangle$ any I , integer I

Under isospin rotation this state behaves like spherical harmonics $Y_l^{m=0}(\theta, \phi)$ under rotation in space around the z -axis

$$\theta \rightarrow \pi - \theta$$

$$\phi \rightarrow \pi - \phi$$

The spherical harmonics $Y_l^m(\theta, \phi) \sim e^{im\phi}$
so $Y_l^{m=0}(\theta, \phi)$ is independent of ϕ

$$Y_l^{m=0}(\theta, \phi) = Y_l^{m=0}(\theta) \sim P_l^0(\cos\theta)$$

$$\text{Under } \theta \rightarrow \pi - \theta, \quad P_l^0(\cos\theta) \rightarrow (-1)^l P_l^0(\cos\theta)$$

Since $|\pi^0\rangle$ is $|1, 0\rangle$, isospin rotation gives a factor of $(-1)^I = -1$

Then since $C|\pi^0\rangle = |\pi^0\rangle$, $\boxed{G = -1}$

Alternatively, we showed in the previous problem that

$$e^{-iAI_2} = \mathbb{1} + \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} (1 - \cos A) + \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \sin A$$

$$e^{i\pi I_2} = e^{-iA I_2} \quad \text{with } A = -\pi, \text{ so}$$

$$e^{i\pi I_2} = \mathbb{1} + \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Ⓢ

Then

$$e^{i\pi I_2} |\pi^0\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -|\pi^0\rangle$$

(b)

$$G|\pi^+\rangle = C e^{i\pi I_2} |\pi^+\rangle = C \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = C \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$G|\pi^+\rangle = C|\pi^+\rangle$$

$$G|\pi^+\rangle = \pm |\pi^+\rangle \quad \left[C|\pi^+\rangle = \pm |\pi^+\rangle \right]$$

$$G|\pi^-\rangle = C \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = C \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

↑ don't know which sign to take -

$$G|\pi^-\rangle = C|\pi^+\rangle = \cancel{C|\pi^+\rangle} = \pm |\pi^-\rangle$$

(c)

$$|\pi^+\rangle = -u\bar{d} \quad |\pi^-\rangle = d\bar{u}$$

$$\text{If } C|u\rangle = |\bar{u}\rangle \quad C|\bar{d}\rangle = |d\rangle$$

$$C|d\rangle = |\bar{d}\rangle \quad C|\bar{u}\rangle = |u\rangle$$

We have

$$G|\pi^+\rangle = C|\pi^-\rangle = C|d\bar{u}\rangle = |u\bar{d}\rangle = -|\pi^+\rangle$$

$$G|\pi^-\rangle = C|\pi^+\rangle = -C|u\bar{d}\rangle = -|d\bar{u}\rangle = -|\pi^-\rangle$$

